STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

WN 28

HOMOGENEITY OF THE HILBERT CUBE

notes taken by Albert Verbeek
from lectures by Prof.dr. J. de Groot

may 1969

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam, The Netherlands.

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conventions: $H = \prod_{n\in\mathbb{N}} [-4^{-n+1}, 4^{-n+1}]$ is the Hilbert cube; $\rho(x,y)=\sqrt{\sum_i (x_i-y_i)^2}$;

$s = \prod_{n\in\mathbb{N}} (-4^{-n+1}, 4^{-n+1})$ is the pseudo-interior;

for $A\subset\mathbb{N}$ $\pi_A: H \to \prod_{n\in A}[-4^{-n+1}, 4^{-n+1}]$ is the projectionmap;

for a space M, Aut M is the group of autohomeomorphisms;

$\mathbb{R}^+$ is the set of positive real numbers.

<u>LEMMA 1.</u> Let M be a compact metric space, $(h_n)_n$ a sequence of autohomeomorphisms of M and $g_n = h_n \circ \ldots h_1 \circ id_H$. Then $\lim_{i\to\infty} g_i$ (defined by pointwise convergence) exists and is an autohomeomorphism if

(i) $\forall m \in M \quad \forall i > n \qquad \rho(m, h_i(m)) < 2^{-n}$

(ii) $\forall m \in M \quad \forall n \qquad \rho(g_{n-1}^{-1}(m), g_n^{-1}(m)) < 2^{-n}$

The straightforward proof is omitted. Notice that (i) $\wedge$ (ii) $\Longleftrightarrow$

$\Longleftrightarrow$ (i) $\wedge$ ($\forall m\in M \quad \forall n \quad \forall i \qquad \rho(g_{n-i}^{-1}(m), g_n^{-1}(m)) < 2^{-n}$).

<u>LEMMA 2.</u> There exists a $g \in$ Aut H and a sequence $(\eta_n)_n$ of positive reals such that for each point $x = (x_n)_n$ from H that satisfies

$$\forall_n \quad 4^{-n+1} - x_n < \eta_n$$

we have

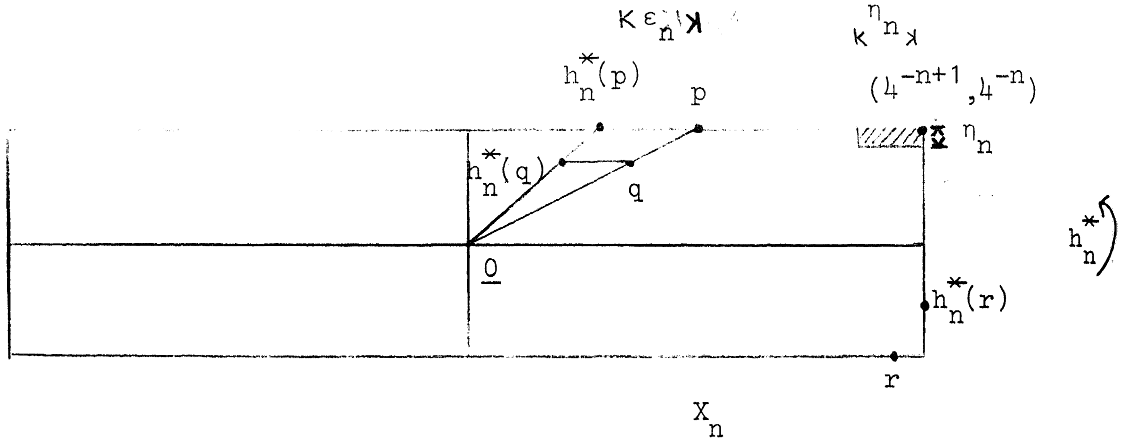
$$g(x) \in s.$$

Proof. We define a sequence $(h_n)_n$ of autohomeomorphisms of H satisfying (i) and (ii) of lemma 1, with M: = H, and put g: = lim g_i. The h_n are defined coordinate-wise:

(iii) $\qquad h_n((x_k)_k) = (x_1, x_2, \ldots x_{n-1}, h_n^*(x_n, x_{n+1}), x_{n+2}, \ldots)$

where $X_n = [-4^{-n+1}, 4^{-n+1}] \times [-4^{-n}, 4^{-n}]$ and $h_n^* \in$ Aut X_n are defined below.

definition of h_n^*:



definition of h_n:

$$\begin{tikzcd} h_n : H \arrow[rr, "h_n"] \arrow[d, phantom, "\cong"] & & H \arrow[d, phantom, "\cong"] \\ \prod_{i=1}^{n-1}[-4^{-n+1},4^{-n+1}] \times X_n \times \prod_{i=n+2}^{\infty}[-4^{-n+1},4^{-n+1}] \arrow[rr, "{(id,h_n^*,id)}"] & & \boxed{} \end{tikzcd}$$

$$\boxed{\prod_{i=1}^{n-1}[-4^{-n+1},4^{-n+1}] \times X_n \times \prod_{i=n+2}^{\infty}[-4^{-n+1},4^{-n+1}]}$$

Firstly we define the restriction of h_n^* to the boundary ∂X_n of X_n (in $\mathbb{R}^2$):
each point of ∂X_n is shifted, counterclockwise, along the boundary, over a distance ε_n (to be specified later), measured along the boundary.
This mapping is extended "linearly" to X_n : if $x \in X_n$ and $\alpha \in \mathbb{R}^+$ such that $\alpha \cdot x$ (scalarmultiplication in $\mathbb{R}^2$) belongs to ∂X_n, then $h_n^*(x) := \frac{1}{\alpha} \cdot h_n^*(\alpha \cdot x)$.
So e.g. $(4^{-n+1},4^{-n})$ and $(\frac{1}{2} \cdot 4^{-n+1}, \frac{1}{2} \cdot 4^{-n} - \frac{\varepsilon}{4})$ are mapped onto $(4^{-n+1}-\varepsilon_n, 4^{-n})$ and $(\frac{1}{2} \cdot 4^{-n+1} - \frac{\varepsilon}{4}, \frac{1}{2} \cdot 4^{-n})$ respectively.
We choose $\varepsilon_n \in \mathbb{R}^+$ small enough to satisfy

(iv) $\qquad \varepsilon_n < 4^{-n}$
and

(v) the distance between two points in X_n with the same first coordinate is multiplied by a factor smaller than 2.

Finally we take $\eta_n := \varepsilon_n / 2$

Notice that (iv) implies condition (i) of lemma 1, and that (v) is equivalent to:

For any two points (x_n, x_{n+1}), (y_n, y_{n+1}) from X_n that satisfy either $x_n = y_n$ or

$$\pi_n\, h_n^*(x_n, x_{n+1}) = \pi_n\, h_n^*(y_n, y_{n+1})$$

we have as well:

(vi)
$$\tfrac{1}{2} < \frac{\rho(h_n^*(x_n, x_{n+1}), h_n^*(y_n, y_{n+1}))}{\rho(\ (x_n, x_{n+1}),\ (y_n, y_{n+1}))} < 2$$

(v)'

as

(vii)
$$\tfrac{1}{2} < \frac{\rho((h_n^*)^{-1}(x_n, x_{n+1}), (h_n^*)^{-1}(y_n, y_{n+1}))}{\rho(\ (x_n, x_{n+1}),\ (y_n, y_{n+1}))} < 2$$

This is seen by using the symmetry of X_n with respect to the origin and $(n+1)^{st}$- axis.

We will next show that condition (ii) of lemma 1 is satisfied:

$$\forall_n \qquad \forall_p \qquad \rho(h_n(p), p) < 4^{-n} \qquad \text{(by (iv))}$$

$$\text{hence } \forall_p \qquad \rho(p, h_n^{-1}(p)) < 4^{-n}$$

Now, by (iii), p and $h_n^{-1}(p)$ have the same (n-1)-coordinate and so

$$\rho(h_{n-1}^{-1}(p), h_{n-1}^{-1}\, h_n^{-1}(p)) < 2 . 4^{-n} \qquad \text{(by (vii))}$$

$$\vdots$$

$$\rho(h_1^{-1} \ldots h_{n-1}^{-1}(p), h_1^{-1} \ldots h_n^{-1}(p)) < 2^{n-1} . 4^{-n} < 2^{-n}$$

$$\text{i.e. } \rho(g_{n-1}^{-1}(p), g_n^{-1}(p)) < 2^{-n} \ (=\text{(ii)}).$$

So by lemma 1 g is an autohomeomorphism of H. It is easily proved by induction to k that for each point $(x_n)_n \in H$ with

$$\forall_n \ [4^{-n+1} - x_n < \eta_n], \text{ it holds that}$$

$$0 < \pi_k\, g_k\, ((x_n)_n) < 4^{-k+1}$$

$$\text{and } 4^{-k+2} - \eta_{k+1} < x_{k+1} \leq \pi_{k+1}\, g_k\, ((x_n)_n) \leq 4^{-k+2}$$

Since $\pi_k\, g_k\, ((x_n)_n) = \pi_k\, g_{k+i}\, ((x_n)_n)$ for all i (bij (iii)), we find

$$\forall_k \ 0 < \pi_k\, g\, ((x_n)_n) < 4^{-k+1}$$

i.e. $\qquad g(x) \in s.$

<u>LEMMA 3a.</u> $\forall p \in [-1,1]\ \forall \eta \in \mathbb{R}^+\ \exists\ \phi^* \in \text{Aut } [-1,1]\quad [1-\phi^*(p) < \eta]$

<u>3b.</u> For each point $x \in H$ and each sequence $(\eta_n)_n$ of positive real numbers there exists a $\phi \in \text{Aut } H$ such that

$$\forall_n \ 4^{-n+1} - \pi_n\, \phi(x) < \eta_n$$

<u>LEMMA 4a.</u> $\forall p \in (-1,1)\ \exists\ \psi^* \in \text{Aut } [-1,1]\quad [\psi^*(p) = 0]$

<u>4b.</u> $\forall y \in s\ \exists\ \psi \in \text{Aut } H\ [\psi(y) = \underline{0}]$

The easy proofs of 3 and 4 are omitted.

<u>THEOREM</u> <u>H is homogeneous.</u>

Proof. We will show: $\forall x \in H\ \exists\ \Phi \in \text{Aut } H\quad \Phi(x) = \underline{0}$.
With the ϕ from lemma 3b and the g from lemma 2 we apply lemma 4b to $y := g\, \phi(x)$ and we find a $\psi \in \text{Aut } H$. Now $\Phi := \psi\, g\, \phi$ is an autohomeomorphism of H which maps x onto $\underline{0}$.

Remarks. It immediately follows that:

Corollary 1. $\prod_{\alpha \in A} [-1,1]$ is homogeneous iff A is infinite.

For $\mathbb{R}^n$ with $n > 2$ the following "strong homogeneity" holds:
For any two ordered k-tuples $(p_1, \ldots p_k)$ and $(q_1, \ldots q_k)$, each consisting of k different points, there is a $\Phi \in \mathrm{Aut}\, \mathbb{R}^n$ such that

$$\Phi(p_i) = q; \quad i = 1, \ldots k.$$

It can easily be seen that also H has this property. The proof uses a slight modification of the lemmas 3a, 3b, 4a and 4b, and finally e.g. a homeomorphism like $\phi^{-1} g^{-1} \psi g \phi$.

In fact we have the following general problem:
If Y is a subspace of X, and $i: Y \subset X$ the canonical embedding, and $f: Y \to X$ an arbitrary embedding, is it possible to extend f to an autohomeomorphism of X?
Eg in the cases $X := \mathbb{R}^2$ and Y is finite, or the cantorset, or the circle, this is possible for arbitrary f. However the Hilbertcube admits wild embeddings of the cantorset, so in this case it is not possible.

Corollary 2. (DE GROOT). For each two points $p, q \in H$ there is a metric d for H and a d-isometry Φ of H such that $\Phi(p) = q$
$\Phi^2 = id_H$.

Proof. We may assume that $p \neq 0$. Let h be the reflection of H which maps $(x_1, x_2, x_3 \ldots)$ onto $(-x_1, x_2, x_3, \ldots)$. By the observation made above there exists a $g \in \mathrm{Aut}\, H$ such that $g(q) = h(p) = (-p_1, p_2, p_3, \ldots)$
$g(p) = p$.

Define $d(x,y) := \rho(gx, gy)$. Now $\Phi := g^{-1} hg$ is a d-isometry of order two, which maps p onto q.

notes made by
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